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Some Left Serial Algebras of Finite Type

G. J. JANUSZ

Department of Mathematics, University of Illinois, Urbana, Illinois 61801

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The representation theory of finite dimensional algebras was studied for a number of years with a view toward solving the Brauer–Thrall conjecture. Roiter's recent work settles this question by showing that an algebra whose indecomposable modules have dimension below a fixed bound has in fact only a finite number of indecomposable modules. His elegant paper [4] is quite elementary and seems almost categorical in spirit. As a result his arguments reveal very little about the structure of algebras with only a finite number of indecomposable modules (hereafter called algebras of *finite type*). We are still left with the problem of recognizing a ring of finite type when one appears.

There are several large classes of algebras known to be of finite type. Semisimple algebras are trivially seen to have finite type. The group algebra $K(G)$ of a finite group G over a field of characteristic p has finite type if and only if the p-Sylow subgroup of G is cyclic [1]. Nakayama [3] showed serial algebras (generalized uniserial in his terminology) were of finite type. Recall that a module is *serial* if it has a unique composition series; equivalently, if every homomorphic image has simple socle. An *algebra* is *serial* if its left and right indecomposable projectives are serial modules. Another large class of algebras of finite type is described in [2]. These are related to the algebras already mentioned but are not included in any of these classes.

If one considers left serial algebras (only the left projective indecomposable modules are serial) then Nakayama's result no longer holds true. For example the algebra of $n \times n$ matrices having arbitrary entries (from a field K) on the diagonal and last row but zeros elsewhere is left serial for all n. However, it has finite type only for $n \leqslant 4$.

In this paper we use the algebras defined in [2] to construct a rather large class of left serial algebras of finite type. Most of these will not be right serial. An example is given in some detail to show for any pair of integers n, r, with $n \geqslant 2$ and $r \geqslant 1$, there is a left serial algebra of finite type which is not right serial, has global dimension n, and has radical N which satisfies $N^r \neq 0$, $N^{r+1} = 0$.

The most difficult part of the job (showing finite type) has been done in [2]. However, there we were interested in group representations so the applications to algebras were not emphasized. Perhaps the most interesting feature of the construction is the combinatorial aspect. A graph is used to describe various objects connected with the algebra; in particular, a multiplication table obtained from the graph is used to define the algebra.

1. The Algebras of Finite Type

We start with a graph $\mathscr{T}$ consisting of edges E, E',... and vertices P, Q,... such that $\mathscr{T}$ is connected and contains no cycles, that is $\mathscr{T}$ is a tree. One vertex is called exceptional and is assigned a multiplicity $m(P) = m$ which is an integer $\geqslant 1$. All other vertices are given the multiplicity $m(Q) = 1$. We assume further that $\mathscr{T}$ is imbedded into the plane in some fixed way. This imposes a cyclic ordering of the edges which contain a given vertex P. Let E_1, E_2, E_3 be distinct edges containing P. We say these edges are in *proper P-order* if E_1, E_2, E_3 is the counterclockwise order of these three edges around P. If the order E_1, E_2, E_3 is clockwise we say the edges are in *improper P-order*.

For a field K an algebra A is constructed from $\mathscr{T}$ in the following way. For each edge E there is a primitive idempotent e_E in A and

$$1 = \sum_{E \in \mathscr{T}} e_E$$

is an orthogonal decomposition of the identity. The Cartan numbers $C_{EE'}$ give the dimension over K of $e_E A e_{E'}$ and are defined by

$$\begin{aligned} C_{EE'} &= 0 && \text{if } E \cap E' = \text{empty}, \\ &= m(P) && \text{if } E \cap E' = P, \\ C_{EE} &= 2 && \text{if the exceptional vertex is not on } E, \\ &= m + 1 && \text{if the exceptional vertex is on } E \\ & && \text{and has multiplicity } m. \end{aligned}$$

Now it is possible to describe a K-basis and multiplication table for A.

(M) (i) For each edge E there is an element X_E such that $e_E A e_E = K[X_E]$ and $X_E^{C-1} \neq 0$,

$$X_E{}^C = 0 \qquad \text{for} \quad C = C_{EE}.$$

(ii) For each pair of edges E, E' with $E \cap E'$ nonempty there is an element $Y_{EE'}$ such that

$$e_E A e_{E'} = K[X_E]\, Y_{EE'}$$

and

$$X_E Y_{EE'} = Y_{EE'} X_{E'}\,.$$

(iii) $\quad Y_{EE'} Y_{E'E} = X_E^{m(Q)} \qquad$ if $E = \overline{PQ}$, $E' = \overline{PQ_0}$, $Q \neq Q_0$,

(iv) $\quad Y_{E_1E_2} Y_{E_2E_3} = Y_{E_1E_3} \qquad$ if E_1, E_2, E_3 are distinct and in improper P-order,

$\qquad = X_{E_1} Y_{E_1E_3} \qquad$ if E_1, E_2, E_3 are distinct, in proper P-order and $m(P) > 1$,

$\qquad = 0 \qquad$ if E_1, E_2, E_3 do not have a common vertex or if they lie in proper P-order and $m(P) = 1$.

Let $E = \overline{PQ}$ be an edge and U_E the projective indecomposable module Ae_E. If N denotes the radical of A, the unique maximal submodule NU_E of U_E has the form

$$NU_E = V_{E,P} + V_{E,Q}\,,$$

where $V_{E,P}$, $V_{E,Q}$ are serial modules. If E' is the edge immediately following E in proper P-order, then

$$\begin{aligned} V_{E,P} &= AY_{E'E} && \text{if} \quad E \neq E' \\ &= AX_E && \text{if} \quad E = E'. \end{aligned}$$

Let F_E denote the simple module U_E/NU_E. Since $V_{E,P}$ is serial, its composition factors occur in a unique order. The order is (from the top down)

$$F_{E_1}, F_{E_2}, \ldots, F_E\,, \tag{*}$$

when E_1, E_2,..., E are the edges of $\mathscr{T}$ on P in proper P-order and $m(P) = 1$. If $m(P) = m > 1$, the sequence (*) is repeated m times (E_1 follows E, etc.).

The two modules $V_{E,P}$, $V_{E,Q}$ have only F_E as a common composition factor and in fact their intersection is simple. Thus in most cases U_E/F_E has a socle of length two. The exception occurs when $V_{E,P}$ or $V_{E,Q}$ is already simple and so equals the socle of U_E. This happens when the edge E is an *end* in the following sense.

Definition. An edge $E = \overline{PQ}$ is an *end* edge if E is the only edge of $\mathscr{T}$ on P and $m(P) = 1$.

Thus U_E is serial if and only if E is an end edge.

2. Left Serial Algebras

Our object is to find left serial algebras. We obtain these as homomorphic images of A. The kernels of these homomorphisms are described in terms of the tree.

(2.1) Let $\mathscr{S}$ be a set of pairs (E, P) such that

(i) E is an edge of $\mathscr{T}$ but not an end edge and P is a vertex of E;

(ii) for any edge $E = \overline{PQ}$ of $\mathscr{T}$ which is not an end edge, exactly one of the pairs (E, P), (E, Q) is in $\mathscr{S}$.

For each such set $\mathscr{S}$ we let $J(\mathscr{S}) = J$ be the two-sided ideal generated by the left ideals $V_{E,P}$ with $(E, P) \in \mathscr{S}$. If it happens that every edge is an end so $\mathscr{S}$ is empty we set $J = (0)$.

Theorem 1. *The algebra A/J is left serial and is an algebra of finite type. If J_0 is an ideal such that A/J_0 is left serial, then $J_0 \supseteq J(\mathscr{S})$ for some set $\mathscr{S}$ satisfying* (2.1).

Proof. The algebra A/J is left serial if and only if the image of every U_E is serial. When E is an end, U_E is already serial so also any homomorphic image will be serial. If $E = \overline{PQ}$ is not an end, then the image of U_E is serial if and only if $V_{E,P}$ or $V_{E,Q}$ is mapped to zero. This is precisely the requirement on J and also shows $J_0 \supseteq J(\mathscr{S})$ for some $\mathscr{S}$.

By Theorem 6.1 of [2] the algebra A has finite type and so any homomorphic image does also.

It will not generally be true that A/J is right serial. The exact situation can be described. First a lemma.

Lemma. *Let $E = \overline{PQ}$ be an edge of $\mathscr{T}$ which is not an end. Then $V_{E,P} \subseteq J$ if and only if $(E, P) \in \mathscr{S}$.*

Proof. Only one direction requires any argument. Suppose $V_{E,P} \subseteq J$. Then in fact

$$V_{E,P} \subseteq Je_E . \qquad (**)$$

The definition of J implies

$$Je_E = \sum_{(E',P') \in \mathscr{S}} V_{E',P'} A e_E .$$

A product $V_{E',P'}Ae_E$ can be nonzero only if E and E' have a common vertex. Moreover, this module may be considered as a sum of homomorphic images of $V_{E',P'}$. Every such image must lie entirely in $V_{E,P}$ or $V_{E,Q}$ because $V_{E',P'}$ is serial.

Now the sum of two submodules of a serial module is equal to the larger of the two. Because of the inclusion (**) it must follow that

$$V_{E,P} \subseteq V_{E',P'}Ae_E\,, \ (E', P') \in \mathscr{S}.$$

We know E is not an end so $V_{E,P}$ has more than one composition factor and the factors correspond to edges on P. It follows that $P = P'$ because the factors of $V_{E'P'}$ correspond to edges on P'. Since the composition lengths of $V_{E,P}$ and $V_{E',P}$ are equal, we must have $V_{E',P} \cong V_{E,P}$ and so $E = E'$. Thus $(E, P) \in \mathscr{S}$ because $(E', P') \in \mathscr{S}$.

Now the condition can be given to insure A/J is also right serial.

THEOREM 2. *The algebra A/J is also right serial if and only if $\mathscr{T}$ and $\mathscr{S}$ satisfy the following condition: For a nonend edge $E = \overline{PQ}$ and suppose $E_i \neq E$, let E_1 immediately precede E in proper P-order and let E_2 immediately precede E in proper Q-order. Then (E_1, P) or (E_2, Q) is in $\mathscr{S}$.*

Remark. The conclusion requires that either E_1 or E_2 is not an end. This shows the condition is very restrictive on $\mathscr{T}$ as well as on $\mathscr{S}$.

Proof. It is necessary to determine the circumstances under which e_EA/e_EJ is serial for each E. When E is an end e_EA is already serial. Suppose $E = \overline{PQ}$ is not an end. Then e_EN is the sum of two (reducible) serial modules

$$e_EN = Y_{EE_1}A + Y_{EE_2}A.$$

In order that e_EA/e_EJ be serial it is necessary and sufficient that either Y_{EE_1} or Y_{EE_2} is in J. This is equivalent to

$$AY_{EE_1} = V_{E_1,P} \subseteq J \qquad \text{or} \qquad AY_{EE_2} = V_{E_2,Q} \subseteq J.$$

By the lemma, (E_1, P) or (E_2, Q) is in $\mathscr{S}$; so the proof is complete.

If by chance $\mathscr{T}$ and $\mathscr{S}$ are selected so that $A/J(\mathscr{S})$ is left and right serial, we can insert one more edge to obtain $\mathscr{T}'$ and a new algebra A' such that $A'/J(\mathscr{S})'$ is not right serial. The new edge need only be attached so it is an end and immediately preceeds some nonend edge. The same $\mathscr{S}$ is used both times.

3. HOMOLOGICAL DIMENSION

We denote $A/J(\mathscr{S})$ by B and abuse the notation slightly by writing e_E to mean an element of A or its image in B.

We consider the question of global dimension. For the algebra A there is little to say because every projective is injective and so gl dim $A = \infty$.

The algebra B can have finite or infinite global dimension. A precise computation can be made but it requires consideration of many cases depending upon the "shape" of $\mathscr{T}$ and the choice of $\mathscr{S}$. The ideas will be illustrated with some representative cases.

PROPOSITION. *Let P be a vertex of $\mathscr{T}$; E_1, E_2 nonend edges which contain P and suppose there is no nonend edge between E_1 and E_2 in proper P-order. Suppose further (E, P) is never in $\mathscr{S}$ for any E. Then* gl dim $B = \infty$.

Proof. Let M be the unique serial module with F_{E_1} as top composition factor and F_{E_0} as bottom factor where E_1, E_0, E_2 are in proper P-order and E_0 immediately preceeds E_2. Allow $E_0 = E_1$ if E_1 and E_2 are consecutive. Then there is an exact sequence

$$0 \to M \to Be_{E_2} \to Be_{E_1} \to M \to 0$$

which shows the projective resolution for M is periodic. Thus M has infinite projective dimension and gl dim $B = \infty$.

It turns out that when $m(P) = 1$ for all P the situation described here is the only one in which B will have infinite global dimension.

As an illustration of a case of finite dimension we start with a graph $\mathscr{T}$ whose vertices are P_0, P_1,..., P_{n+r} and whose edges are

$$\begin{aligned} E_i &= \overline{P_{i-1}P_i} \qquad 1 \leqslant i \leqslant n, \quad n \geqslant 2, \\ E_{n+j} &= \overline{P_nP_{n+j}} \qquad 1 \leqslant j \leqslant r. \end{aligned} \tag{3.1}$$

Increasing subscripts determine proper P_n-order. Set $m(P) = 1$ for all P. The nonends are E_2,..., E_n. For $\mathscr{S}$ we use the set

$$\{(E_2, P_2), (E_3, P_3),..., (E_n, P_n)\}.$$

Let B_i denote the image of Ae_{E_i} in $B = A/J(\mathscr{S})$ and let F_i denote the simple homomorphic image of B_i.

We shall use the notation

$$B_i \sim (i, j, k,...)$$

to mean

$$B_i/NB_i \cong F_i, \qquad NB_i/N^2B_i \cong F_j, ...$$

We have then

$$\begin{aligned} B_1 &\sim (1, 2, 1), \\ B_2 &\sim (2, 1), \\ &\cdots \\ B_k &\sim (k, k-1), \qquad 2 \leqslant k \leqslant n, \\ B_{n+1} &\sim (n+1, n+2, \ldots, n+r, n), \\ B_{n+2} &\sim (n+2, \ldots, n+r, n), \\ &\cdots \\ B_{n+r} &\sim (n+r, n). \end{aligned}$$

Notice that

$$0 \to B_{n+j+1} \to B_{n+j} \to F_{n+j} \to 0$$

is exact for $1 \leqslant j < r$ so $h \dim(F_{n+j}) = 1$. Now for $1 < j \leqslant n$, we find

$$0 \to B_2 \to B_1 \to B_2 \to B_3 \to \cdots \to B_{j-1} \to B_j \to F_j \to 0$$

is a minimal resolution; so

$$h \dim(F_j) = j, \qquad 1 \leqslant j \leqslant n. \tag{*}$$

For $j = 1$, we have

$$0 \to B_2 \to B_1 \to F_1 \to 0;$$

so (*) holds also for $j = 1$.

Finally,

$$0 \to F_n \to B_{n+r} \to F_{n+r} \to 0$$

is exact; so

$$h \dim(F_{n+r}) = 1 + h \dim(F_n) = n + 1.$$

The global dimension of B (or any Artinian ring) is the maximum of the homological dimensions of the simple modules. In this case

$$\operatorname{gl} \dim(B) = n + 1.$$

The reader may find it amusing to change this example by using the same $\mathscr{T}$ but letting $\mathscr{S}$ consist of all pairs (E_i, P_{i-1}) for $2 \leqslant i \leqslant n$. It turns out then

$$\operatorname{gl} \dim(A/J(\mathscr{S})) = n + 2r - 1.$$

In either case given here, the algebras are not right serial. The test given in Theorem 2 is easily applied.

It was required that $n \geqslant 2$, so these do not provide examples with global dimension one or two. For completeness we mention such examples. Take $\mathscr{T}$ as in (3.1) with $n = 2$ and $\mathscr{S}$ as first defined with pairs (E_i, P_i).

The socle of $B_1 \oplus B_2$ is an ideal J_1 in B and the socle of $B_1 \oplus B_2 \oplus B_3$ is an ideal J_2 in B. Then

$$\operatorname{gl}\dim(B/J_1) = 1,$$
$$\operatorname{gl}\dim(B/J_2) = 2.$$

We remark that $A/J(\mathscr{S})$ never has global dimension one. Any algebra with global dimension one has a simple projective module. However, NU_E is never in $J(\mathscr{S})$ so F_E cannot be projective over B.

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